(ii) Let $\alpha \in F$, and $w \in W$. Again, since T is an isomorphism, there exists $v \in V$ s.t. $T(v) = w$, which means that $v = T^{-1}(w)$. But then:

$T(\alpha v) = \alpha T(v)$; since T is linear

$T^{-1}(T(\alpha v)) = T^{-1}(\alpha T(v))$; apply T^{-1} to both sides

$\alpha v = T^{-1}(\alpha T(v))$; but by definitions ; $v = T^{-1}(w) \wedge T(v) = w$

$\boxed{\alpha T^{-1}(w) = T^{-1}(\alpha w)}$

(i) & (ii) $\Rightarrow T^{-1}$ is a linear transformation from W to V. +10

(b) Now, let $A \in M_n(F)$ and $L_A : F^n \to F^n$ denote the linear transformation given by left multiplication by A. Prove that L_A is an isomorphism $\Leftrightarrow$ the matrix A is invertible.

Pf: ($\Rightarrow$) Suppose L_A is an isomorphism. We want to find a matrix B s.t: $AB = BA = I$, and conclude that A is invertible with inverse B.

By part (a); since LA is an isomorphism and a linear transformation, L_A^{-1} (inverse of LA) is also a linear transformation. We proved in class that for each linear transformation T, there is a unique $M \in M_n(F)$ s.t $T = L_M$. Apply this to L_A^{-1} to get that $L_A^{-1} = L_B$, for some $B \in M_n(F)$. But then B is the inverse of A since for all $\vec{v} \in F^n$

$$(A \cdot B)(\vec{v}) = A(B(\vec{v})) = A(L_A^{-1}(\vec{v})) = LA(LA^{-1}(\vec{v})) = (LA \circ LA^{-1})(\vec{v}) = \vec{v}$$
$$(B \cdot A)(\vec{v}) = B(A(\vec{v})) = B(LA(\vec{v})) = LA^{-1}(LA(\vec{v})) = (LA^{-1} \circ LA)(\vec{v}) = \vec{v}$$

$\Rightarrow A \cdot B = B \cdot A = I \Rightarrow \boxed{A^{-1} = B}$; A is invertible.

($\Leftarrow$) Suppose $A \in M_n(F)$ is invertible. We want to find a linear transformation $T : F^n \to F^n$ s.T $T \circ L_A = L_A \circ T = id$; i.e., T is the inverse of L_A. Since A is invertible, it makes sense to consider $L_{A^{-1}}$ = left multiplication by A^{-1}. We know this is a unique l.t. and moreover,

$$(L_A \circ L_{A^{-1}})(\vec{v}) = LA(L_{A^{-1}}(v)) = A(A^{-1}(v)) = (AA^{-1})v = Iv = v$$
$$(LA \circ LA^{-1})(\vec{v}) = L_A^{-1}(LA(v)) = A^{-1}(A(v)) = (A^{-1}A)v = Iv = v$$

Hence, $T = L_{A^{-1}}$ is the inverse of $LA \Rightarrow L_A$ is 1-1, onto, l.t $\Rightarrow LA$ is an isomorphism

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(6) Let V be an F-vector space and let W be a subspace.

(a) Prove that V is finite dimensional $\Leftrightarrow$ W and V/W are finite dimensi

Pf: ($\Rightarrow$) Suppose that V is finite dimensional. We proved in class that any subspace of a finite dimensional v.s. is also finite dimensional, so W is finite dimensional. To show that V/W is finite dimensional, let us construct a finite basis for it.

Let $\{w_1,\ldots,w_m\}$ be a basis for W and $\{w_1,\ldots,w_m,v_1,\ldots,v_n\}$ a basis for V.

Claim: $\{v_1+W,\ldots,v_n+W\}$ is a basis for V/W

Pf: We need to show that (i) $\{v_1+W,\ldots,v_n+W\}$ is a linearly independent se and (ii) $Span(\{v_1+W,\ldots,v_n+W\}) = V/W$.

(i) Let $\alpha_1,\ldots,\alpha_n \in F$. then, $\alpha_1(v_1+W)+\cdots+\alpha_n(v_n+W) = 0+W = W$ $\Rightarrow$ by def o + on V/W: $(\alpha_1v_1+\cdots+\alpha_nv_n)+W = W \Rightarrow \alpha_1v_1+\cdots+\alpha_nv_n \in W$, so this elemen can be written uniquely as a linear combination of the basis of W: let $\beta_1,\ldots,\beta_m$

$$\alpha_1v_1+\cdots+\alpha_nv_n = \beta_1w_1+\cdots+\beta_mw_n,$$

$$\alpha_1v_1+\cdots+\alpha_nv_n - \beta_1w_1-\cdots-\beta_mw_n = 0 \text{ ; but the LHS} \in V \text{ and}$$

$\{v_1,\ldots,v_n,w_1,\ldots,w_m\}$ is a basis for $V \Rightarrow \alpha_i = 0; \beta_j = 0 \;\forall i,j \Rightarrow \boxed{\alpha_i = 0} \;\forall i.$

(ii) Let $a+W \in V/W$. By definition $a\in V$, and so it can be written as:

+10

$$a = \alpha_1w_1+\cdots+\alpha_mw_m+\beta_1v_1+\cdots+\beta_nv_n.$$

$$\begin{aligned}
a+W &= (\alpha_1w_1+\cdots+\alpha_mw_m+\beta_1v_1+\cdots+\beta_nv_n)+W \text{ , but } \alpha_1w_1+\cdots+\alpha_mw_m \\
&= (\alpha_1w_1+\cdots+\alpha_mw_m)+W + (\beta_1v_1+\cdots+\beta_nv_n)+W \\
&= (0+W) + (\beta_1v_1+\cdots+\beta_nv_n)+W \\
&= \beta_1v_1+\cdots+\beta_nv_n W \\
&= (\beta_1v_1+W)+(\beta_2v_2+W)+\cdots+(\beta_nv_n+W). \\
&= \beta_1(v_1+W)+\beta_2(v_2+W)+\cdots+\beta_n(v_n+W). \Rightarrow a+W \in Span(\{v_1+W,\ldots
\end{aligned}$$

(i) and (ii) $\Rightarrow$ $\{v_1+W,\ldots,v_n+W\}$ is a basis for V/W

$\Rightarrow \dim(V/W) = n$

$\Rightarrow V/W$ is finite dimensional

M403 - Fall 2013 - HW7 - Enrique Areyan (7)

(6)(a) (⇐) Suppose W and V/W are finite dimensional. Since W is finite dimensional, we have a finite base for this space, say, $\{w_1,\dots,w_n\}$. Likewise, V/W has a finite basis, say $\{v_1+W,\dots,v_n+W\}$ for $v_1,\dots,v_n \in V$. Following a reasoning similar to that of (⇒), a basis for V is $\{w_1,\dots,w_m,v_1,\dots,v_n\}$. this proof is very similar to the one for (⇒), i.e., we have to prove (i) $\{w_1,\dots,w_m,v_1,\dots,v_n\}$ is linearly independent and (ii) $V = span(\{w_1,\dots,w_m,v_1,\dots,v_n\})$.

(i) $\alpha_1 w_1 + \dots + \alpha_m w_m + \beta_1 v_1 + \dots + \beta_n v_n = 0$. Since $v_i + W_i \neq 0 + W_i \Rightarrow v_i \notin W, \forall$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_m = \beta_1 = \beta_2 = \dots = \beta_n = 0$.

(ii) Let $v \in V$. Suppose v can not be written as a linear combinatio[n] of $\{w_1,\dots,w_m,v_1,\dots,v_n\}$. So, v cannot be written as a linear combina[tion] of $\{w_1,\dots,w_m\}$ and cannot be written as a l.c. of $\{v_1,\dots,v_n\}$.
$\Rightarrow v \notin W$. and $v \notin V/W$. but for all $v \in V \Rightarrow v \in W$ or $v \in V/W$, by taking the cannonical map $\pi(v) = v+W$. therefore $v \notin V$, a contr[adiction]

So $v \in V \Rightarrow v \in span(\{w_1,\dots,w_m,v_1,\dots,v_n\})$.

(i)&(ii) $\Rightarrow \{w_1,\dots,w_m,v_1,\dots,v_n\}$ is a basis of V

$\Rightarrow dim(V) = n+m$

$\Rightarrow V$ is finite dimensional

(6)(b) Now assume V is finite dimensional and prove that

$$dim(W) + dim(V/W) = dim(V).$$

Pf: the proof follows from the construction of a basis for $V/$[W] done in ((6)(a)(⇒)). there we found that if

$\{w_1,\dots,w_m\}$ is a basis for W and

$\{w_1,\dots,w_m,v_1,\dots,v_n\}$ is a basis for V. then

$\{v_1+W,\dots,v_n+W\}$ is a basis for W. counting the ele[ments] of each basis we have that: $m+n = dim(V)$; $m = dim(W)$, $n = dim(V/W)$

$\Rightarrow \boxed{dim(W) + dim(V/W) = dim(V)}$

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(7) Let V and U be vector spaces over a field F.
Let $T: V \to U$ be a linear transformation.

(a) Prove that $T(V) = \{T(v) | v \in V\}$ is a subspace of W and is finite dimensional if V is finite dimensional.

Pf: (i) $T(V)$ is a subspace since

(I) $T(V)$ is closed under $+$ and scalar multiplication

Let $x \in T(v)$ and $y \in T(V)$. then $x = T(\vec{v_1})$ for some $\vec{v_1} \in V$ and $y = T(\vec{v_2})$ for some $\vec{v_2} \in V$.

But then

$x + y = T(\vec{v_1}) + T(\vec{v_2}) = T(\vec{v_1} + \vec{v_2})$ since T is linear.

Hence, there exists $z \in V : z = \vec{v_1} + \vec{v_2}$ s.t. $x + y = T(\vec{z})$, which means that $x+y \in T(V)$.

Let $\alpha \in F$ and $x \in T(v)$. then $x = T(\vec{v_1})$, for some $\vec{v_1} \in V$. But then

$\alpha x = \alpha T(\vec{v_1}) = T(\alpha \vec{v_1})$, since T is linear.

Hence, $\alpha x \in T(V)$; just choose $\alpha \vec{v_1} \in V$ s.t. $T(\alpha \vec{v_1}) = \alpha T(\vec{v_1}) = \alpha$

(II) $0 \in T(V)$ this is because T is a linear transformation, so tha

$T(0_v) = 0_U \in U$.

(I) & (II) $\Rightarrow$ $T(V)$ is a subspace of U.

(ii) If V is finite dimensional then $T(V)$ is finite dimensio

Suppose V is finite dimensional and let $\{v_1, \ldots, v_n\}$ be a basis for V

then, $\{T(v_1), \ldots, T(v_n)\}$ is either a basis for the space $T(V)$

or a linearly independent set. which we can use to form a basis

for $T(V)$ by removing some $T(v_i$'s). In either case $\dim(T(V))$

finite. In fact, $\dim(T(V)) \leq n = \dim(V)$.

Let $\alpha_1, \ldots, \alpha_n \in F$. Look at $\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0 = T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$.

If $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$; since $\{v_1, \ldots, v_n\}$ is a base for V.

therefore, $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent. Moreover, in this ca

$y \in T(V)$ can be written as $T(\beta_1 v_1 + \cdots + \beta_n v_n) = T(v) = y$. Uniquely so

$\text{span}(\{T(v_1), \ldots, T(v_n)\}) = T(V) \Rightarrow \dim(T(V)) = n$.

otherwise, $\alpha_1 v_1 + \cdots + \alpha_n v_n \neq 0 \Rightarrow \alpha_1 v_1 + \cdots + \alpha_n v_n \in Ker(T)$. In this case not all α_i can be zero so $\{T(v_1), \ldots, T(v_n)\}$ is a linearly dependent set. Without loss of generality, suppose $T(v_1) = \gamma_1 T(v_2) + \cdots + \gamma_n T(v_n)$. Apply the same reasoning to $\{T(v_2), \ldots, T(v_n)\}$. Having done this, at some finite number of steps $\{T(v_k), \ldots, T(v_n)\}$ will be either empty, in which case T is the trivial (null) transformation and so $\dim(T(V)) = 0$ OR $\{T(v_k), \ldots, T(v_n)\}$ will form a linearly independent set which clearly spans $T(V)$. therefore, $\dim(T(V)) \leq n$, finite.

(b) Prove that if V is finite dimensional then

$$\dim(Ker(T)) + \dim(T(V)) = \dim(V).$$

Pf: By fundamental theorem:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & U \\ Ker(T) = V/M \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Moreover, we know that any subspace of V is just the kernel of a linear transformation

By exercise (6), since V is finite dimensional, so is $Ker(T)$. By part (a), $T(V)$ is also finite dimensional, so the equation we want to prove is at least well defined (makes sense).

Moreover in exercise (6) we proved:

$$\dim(W) + \dim(V/W) = \dim(V)$$

Considering $W = T(V)$ (the image of T), then the result follows

$$\dim(Ker(T(V)) + \dim(T(V)) = \dim(V).$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & T(V) \\ Ker(T) \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

We proved in (a) that $T(V) \leq U$.